MEMO – GRADE 12 AP EXAM

JULY EXAM 2014

QUESTION 1:

$a^n - b^n$ is divisible by $a-b$

Prove true for $n=1$ ✓

$a^1 - b^1 = a - b$ ✓

$\therefore$ True for $n=1$

(12)

Assume true for $n=k$ ✓

$a^k - b^k = p(a-b)$ ✓ ✓

Prove true for $n=k+1$ ✓

$a^{k+1} - b^{k+1}$ ✓

$= a^k \cdot a - b^k \cdot b$

$= (p(a-b) + b^k)a - b^k \cdot b$ ✓

$= pa(a-b) + b^k \cdot a - b^k \cdot b$

$= pa(a-b) + b^k(a-b)$ ✓

$= (pa + b^k)(a-b)$ ✓

It is true for $n=1$ and for $n=k+1$, therefore it is true for $n \in \mathbb{N}$ by mathematical induction ✓ ✓

QUESTION 2

a) $x^4 + 6x^3 + px^2 - 12x - 20 = 0$

$x = -3 - i$

$x = -3 + i$ ✓

$x^2 + 6x + 10 = 0$ ✓✓

$(x^2 + 6x + 10)(x^2 + ax - 2) = 0$

$-12x + 10ax = -12x$

$10a = 0$

$\underline{\underline{a = 0}}$

(7)

$(x^2 - 2) = 0$

$\underline{\underline{x = \pm\sqrt{2}}}$ ✓✓

$\underline{\underline{p = 8}}$ ✓✓

b) $\operatorname{Im}(-3+13i)(3-7i)$ ✓

$= (-9 + 21i + 39i - 91i^2)$

$= 82 + 60i$ ✓✓

(4)

$\therefore \operatorname{Im}(82+60i) = \underline{60}$ ✓

2) $\dfrac{3-7i}{-3+13i} \times \dfrac{-3-13i}{-3-13i}$ ✓

$= \dfrac{-9-39i+21i+91i^2}{9-169i^2}$ ✓ ✓

(6)

$= \dfrac{-100-18i}{178}$ ✓

$= \underline{\underline{\dfrac{-50}{89} - \dfrac{9}{89}i}}$ ✓✓

C) 1. $|\ln(x+1)| - 2 \leq 0$

$\ln(x+1) = 2$

$x+1 = e^2$

$x = e^2 - 1$ (6,4)

$-\ln(x+1) = 2$

$x+1 = e^{-2}$

$x = e^{-2} - 1$

6

$e^{-2} - 1 \leq x \leq e^2 - 1$

$-0,86 \leq x \leq 6,4$

2) $e^{2x} + e^x = 3e^x + 2^3$

$e^{2x} - 2e^x - 8 = 0$

$e^x = 4 \quad e^x \neq -2$

6

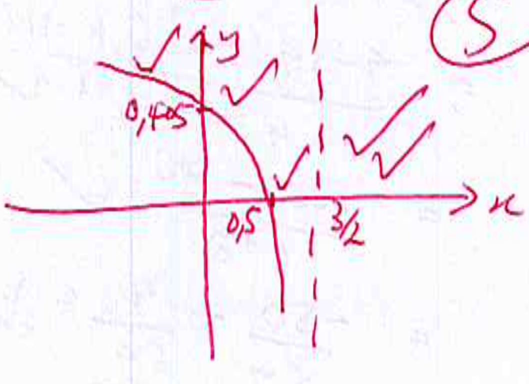
$x = \ln 4$

4) $f(x) = -\ln\left(\frac{2}{3-2x}\right)$

$= \ln\left(\frac{2}{3-2x}\right)^{-1}$

$= \ln\left(\frac{3-2x}{2}\right)$

5




2) $y = \ln\left(\frac{3}{2} - x\right)$

$x = \ln\left(\frac{3}{2} - y\right)$

$e^x = \frac{3}{2} - y$

4

$y = \frac{3}{2} - e^x$

$f^{-1}(x) = \frac{3}{2} - e^x$

QUESTION 3

$f(g(x)) = 2x^2 + 3x - 2$

$g'(2) = 5 \quad g(2) = 4$

a) $f(4) = f(g(2))$

$= 8 + 6 - 2$

4

$= 12$

b) $f'(4) = f'(g(x)) \cdot g'(x)$

$= 4x + 3$

$= f'(g(2)) \cdot g'(2)$

$f'(4) \cdot 5 = 11$

6

$f'(4) = 11/5$

QUESTION 4

$$g(x) = \begin{cases} 4 & x \le 0 \\ -x^2+4 & 0 < x \le 2 \\ & x > 2 \end{cases}$$

a) 1. $\lim_{x \to 0} 4 = 4$ (3)

$\lim_{x \to 0} -x^2+4 = 4$

$f(0) = 4 = \lim_{x \to 0} f(x)$

∴ is continuous.

2) $\lim_{x \to 0^+} f'(x) = \lim_{x \to 0} 0 = 0$

$\lim_{x \to 0^+} (-2x) = 0$ (4)

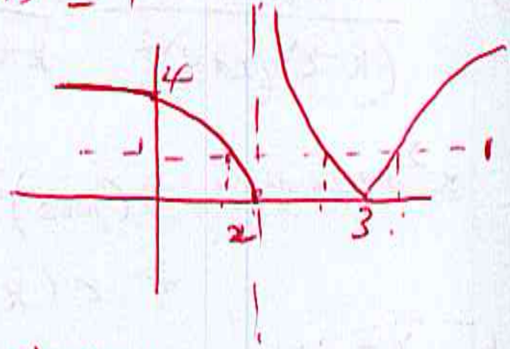
∴ it is differentiable.

b) $\lim_{x \to 2^-} (-x^2+4) = 0$

$\lim_{x \to 2^+} |\ln(x-2)| = \ln 0$ (undefined) (6)

∴ vertical asymptote at $x=2$ (jump discontinuity)

c) $g(x) \le 1$



$-x^2+4 = 1$
$x = \sqrt{3}$

$x \in [\sqrt{3} : 2]$

$\ln(x-2) = 1$
$x = e+2$

$-\ln(x-2) = 1$
$x = 2+e^{-1}$

or $x \in [2+e^{-1} : e+2]$ (4)

$[2{,}4 : 4.7]$

QUESTION 5

a1. $\lim_{x \to 0} \frac{3\sin x - \sin 3x}{3x}$

$= \lim_{x \to 0} \frac{3\cos x - 3\cos 3x}{3}$ (6)

$= \underline{0}$

2) $\lim_{x \to \infty} (\sqrt{x^2+2x} - x) \times \left(\frac{\sqrt{x^2+2x}+x}{\sqrt{x^2+2x}+x}\right)$

$= \lim_{x \to \infty} \frac{x^2+2x-x^2}{\sqrt{x^2+2x}+x}$ (6)

$= \lim_{x \to \infty} \frac{2x/x}{\sqrt{\frac{x^2}{x^2}+\frac{2x}{x^2}}+\frac{x}{x}} = \frac{2}{2} = \underline{1}$

b) $\frac{x^2+14x+18}{(x-2)(x+3)^2} = \frac{A}{x-2} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$

$x^2+14x+18 = A(x+3)^2 + B(x-2)(x+3) + C(x-2)$

Sub $x=2$

$A = 2$

Sub $x=-3$ $\quad C = 3$ $\quad$ 8

Sub $x=0$ $\quad 18 = 18 - 6B - 2C$

$B = -1$

$= \frac{2}{x-2} - \frac{1}{x+3} + \frac{3}{(x+3)^2}$

c) $f(x) = \frac{1}{(3-x^2)^{17}} = (3-x^2)^{-17}$ $\quad$ 10

$f'(x) = -17(3-x^2)^{-18} \cdot -2x$

$f''(x) = -17 \cdot -18 \cdot (3-x^2)^{-19} \cdot -2x \cdot -2x$

$f'''(x) = -17 \cdot -18 \cdot -19 \, (3-x^2)^{-20} \cdot (-2x)(-2x)^2$

$f^n(x) = (-1)^n \frac{(16+n)!}{16!} (3-x^2)^{-17-n} \cdot (-2x)^n$

d) $3y^4 + 4x - x^2\sin y - 4 = 0$

$12y^3\frac{dy}{dx} + 4 - 2x\sin y - x^2\cos y \frac{dy}{dx} = 0$

$\frac{dy}{dx}(12y^3 - x^2\cos y) = 2x\sin y - 4$

$\frac{dy}{dx} = \frac{2x\sin y - 4}{12y^3 - x^2\cos y}$

Sub $(1,0)$ $\quad \frac{dy}{dx} = 4$.

$\therefore y = -\frac{1}{4}x + c$ $\quad$ 12

$0 = -\frac{1}{4}(1) + c$

$y = -\frac{1}{4}x + \frac{1}{4}$

3) $\int_0^{\pi/3} (\sin^3 x \cos^2 x)\, dx$

$= \int_0^{\pi/3} \sin^2 x \cdot \sin x \cos^2 x \, dx$

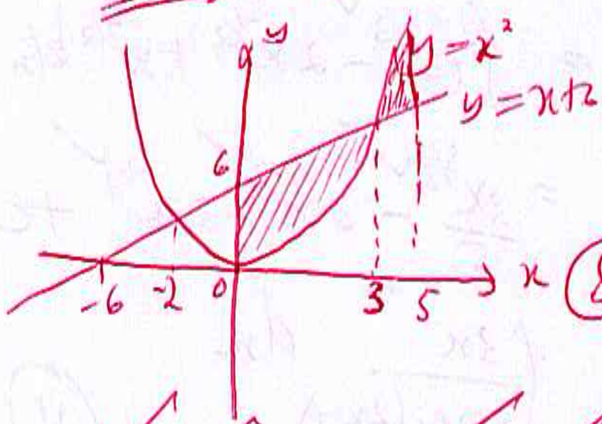
$= \int_0^{\pi/3} (1-\cos^2 x) \sin x \cdot \cos^2 x \, dx$

$= \int_0^{\pi/3} (\sin x \cos^2 x - \sin x \cos^4 x)\, dx$

$= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x \Big|_0^{\pi/3}$ (10)

$= \underline{\underline{47/480}}$ $\underline{\underline{(0,048)}}$

b)




(8)

$\int_0^3 (x+2 - x^2)\, dx + \int_3^5 (x^2 - (x+2))\, dx$

$= \frac{x^2}{2} + 2x - \frac{x^3}{3}\Big|_0^3 + \frac{x^3}{3} - \frac{x^2}{2} - 2x \Big|_3^5$

$= \frac{157}{6} = \underline{\underline{26,17}}$

c) $x = \frac{2}{3}(y-1)^{3/2} \quad 1 \le y \le 4$

$\text{length} = \int_1^4 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dx$

$\frac{dx}{dy} = \frac{2}{3} \times \frac{3}{2} (y-1)^{1/2} \cdot 1$

$= \underline{\underline{(y-1)^{1/2}}}$

$\text{length} = \int_1^4 \sqrt{1 + ((y-1)^{1/2})^2}\, dy$

$= \int_1^4 \sqrt{y}\, dy$ (8)

$= \frac{2y^{3/2}}{3}\Big|_1^4 = \underline{\underline{\frac{14}{3}}}$ (4,7)

d) $V = \pi \int_a^b (f(x))^2 dx$

$= \pi \int_1^2 (\sqrt{x-1} - (x-1)^2)^2 dx$

$= \pi \int_1^2 ((x-1)^{1/2} - (x-1)^2)^2 dx$

$= \pi \int_1^2 ((x-1) - 2(x-1)^{5/2} + (x-1)^4)\, dx$

$= \pi \left[ \frac{(x-1)^2}{2} - \frac{4(x-1)^{7/2}}{7} + \frac{(x-1)^5}{5} \right|_1^2$

$= \underline{\underline{0,404}}$ $\underline{\underline{\frac{9}{70}\pi}}$ (9)

QUESTION 6

$$f(x) = \frac{x^3+4x^2+x-6}{x^2-1}$$

$$= \frac{(x-1)(x+2)(x+3)}{(x-1)(x+1)}$$

$$= \frac{(x+2)(x+3)}{x+1}$$

a) x intercepts:

$x = -2$ or $x = -3$

y intercept: $y = 6$

Stationary points:

$$f'(x) = \frac{(x+1)\cdot(2x+5)-(x+2)(x+3)\cdot 1}{(x+1)^2}$$

(9)

$x = -1 \pm \sqrt{2}$

$0{,}414$ or $-2{,}414$

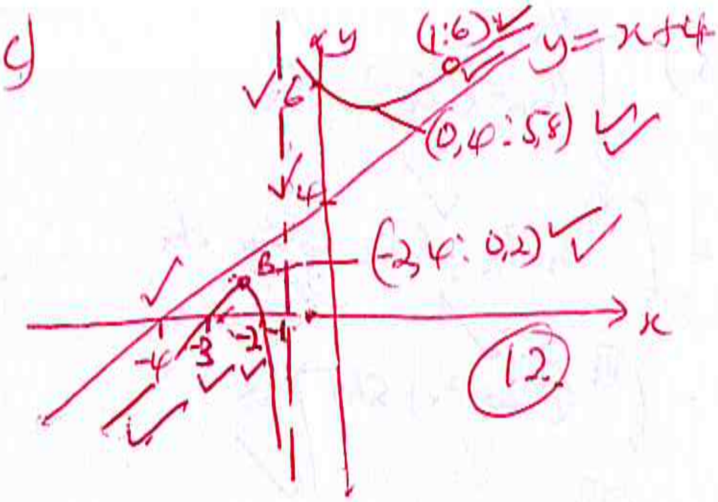
$(0{,}414;\ 5{,}828)$ ; $(-2{,}414;\ 0{,}172)$

b) VA: $x = -1$ (6)

Removable discont: $x = 1$. $y = 6$.

Oblique: $y = x+4$

c)




QUESTION 7

a1. $\int \left(x^{1/3} - x^{-2/3}\right)^2 dx$ (6)

$= \int \left(x^{2/3} - 2x^{-1/3} + x^{-4/3}\right) dx$

$= \frac{3x^{5/3}}{5} - 3x^{2/3} - 3x^{-1/3} + c$

2) $\int \frac{3x}{\operatorname{cosec}(2x-1)} dx$ (11)

$= \int 3x \sin(2x-1)\, dx$

$= 3 \int x \sin(2x-1)\, dx$

$f(x) = x$ $g'(x) = \sin(2x-1)$

$f'(x) = 1$ $g(x) = -\frac{\cos(2x-1)}{2}$

$= 3\left[-\frac{x\cos(2x-1)}{2} + \int \frac{\cos(2x-1)}{2} dx\right]$

$= 3\left[-\frac{x\cos(2x-1)}{2} + \frac{\sin(2x-1)}{4}\right] + c$

QUESTION 4

$\sum x = 1800$ $\sum y = 36$

$\sum xy = 6438,6$ $\sum x^2 = 336296$

$\sum y^2 = 126,34$

a) $y = a + bx$

$$b = \frac{12 \cdot (6438,6) - (1800)(36)}{12(336296) - (1800)^2}$$

$= 0,0157$

$\bar{x} = 150$ $\bar{y} = 3$

$3 = a + 0,0157(150)$ (8)

$a = 0,6501$

$\therefore y = 0,6501 + 0,0157x$

b) $y = 0,6501 + 0,0157(200)$

$= 3,7901$ (2)

c) $y = 0,6501 + 0,0157(1500)$

$= 24,2001$

⇒ extrapolation - out of the domain

⇒ outlier (3)

d) $$r = \frac{12(6438,6) - 1800 \cdot (36)}{\sqrt{(12(336296) - (1800)^2)(12(126,34) - (36)^2)}}$$

$= 0,9419$ (6)

e) there is strong positive linear relationship between number of days and distance traveled. (2)

QUESTION 5

i) $\bar{x} \pm z_{score} \frac{\delta}{\sqrt{n}}$ $n = 30$

$\bar{x} = 204,933$

$\delta = 11,5612$

$204,9333 \pm 1,96 \cdot \frac{11,5612}{\sqrt{30}}$

$[200,7162 \quad 209,0704]$ (9)

b) $\frac{6}{30} \pm 1,96 \cdot \sqrt{\frac{\frac{6}{30} \cdot \frac{24}{30}}{30}}$ (9)

$[0,0569 ; 0,3431]$

SECTION B

QUESTION 1

1 2 B 4 5 6 D

$$\frac{5\times 4 \quad 3\times 2\times 1}{3!}$$

$= \underline{20}$

5

QUESTION 2

$$f(x) = \begin{cases} cx^2 & 0 \le x \le 2 \\ 2c(4-x) & 2 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

a) $\int_0^2 cx^2\,dx + \int_2^4 2c(4-x)\,dx = 1$

$\left.\frac{cx^3}{3}\right|_0^2 + \left. 8cx - cx^2 \right|_2^4 = 1$

$\frac{8c}{3} + (32c - 16c) - (16c - 4c) = 1$

8

$\underline{c = 0{,}15}$

b) $\int_0^2 cx^2\,dx + \int_2^m (8c - 2cx)\,dx = \frac{1}{2}$

$\left.\frac{0{,}15x^3}{3}\right|_0^2 + \left. 1{,}2x - \frac{0{,}3x^2}{2}\right|_2^m = \frac{1}{2}$ 8

$0{,}4 + (1{,}2m - 0{,}15m^2) - (1{,}8) = \frac{1}{2}$

$-0{,}15m^2 + 1{,}2m - 1{,}9 = 0$

$\underline{m = 2{,}1742}$

c) median $= \underline{2{,}1742}$ 2

QUESTION 3

a) $\frac{{}^3C_1\,{}^6C_3}{{}^9C_4} + \frac{{}^3C_2\,{}^6C_2}{{}^9C_4} + \frac{{}^3C_3\,{}^6C_1}{{}^9C_4}$

$= \underline{37/42}$ (0,8809) 8

or $1 - \frac{{}^3C_0\,{}^6C_4}{{}^9C_4} = \underline{37/42}$

b) $P(N = n) = {}^{10}C_n\left(\frac{2}{3}\right)^n\left(\frac{1}{3}\right)^{10-n}$

$n = 0; 1; 2 \ldots ; 10$

1) $P(\text{success}) = 2/3$ 3

2) $P(N \le 4)$

$= P(N=0) + P(N=1) + P(N=2) + P(N=3) + P(N=4)$ 7

$= {}^{10}C_0\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^{10} + {}^{10}C_1\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^9$

$+ {}^{10}C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^8 + {}^{10}C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^7$

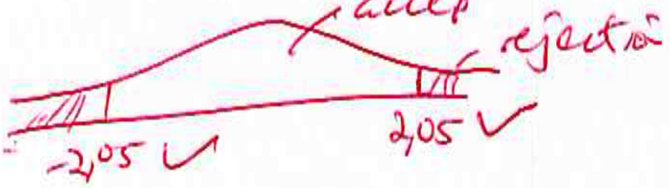
$+ {}^{10}C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^6$

$= \frac{1507}{19683} = \underline{0{,}0766}$

QUESTION 6

i) $H_0: \mu = 8.00$

$H_1: \mu \neq 8.00$




$$Z = \frac{8{,}011 - 8{,}00}{\sqrt{\frac{0{,}0001}{9}}}$$

9

$= \underline{3{,}3}$

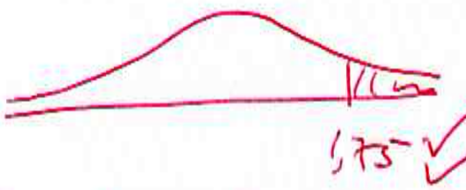
There is enough evidence to reject $H_0$ at 4% significance level and conclude that there is a change in the mean length of screw drivers.

ii) $H_0: \mu_x = \mu_y$

$H_1: \mu_x > \mu_y$

$x \Rightarrow$ Batch A

$y \Rightarrow$ Batch B.




14

$$Z = \frac{8{,}011 - 7{,}992}{\sqrt{\frac{0{,}0001}{9} + \frac{0{,}0001}{16}}} = \underline{\underline{4{,}56}}$$

- there is enough evidence to reject $H_0$ at 4% sig level and conclude that population (B) has smaller mean